以伯恩斯坦多項式直接法求解變分問題

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摘　要

本論文是以伯恩斯坦多項式，對變分問題建立一個清楚的求解過程。變分問題以伯恩斯坦多項式直接法求解，並簡化成代數方程式的求解。伯恩斯坦多項式的特性完全應用於計算過程。三個說明的例子包括在論文裡。

關鍵字：伯恩斯坦多項式，變分問題，直接法

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Bernstein Polynomials Direct Method for Solving Variational Problems

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Abstract

This paper establishes a clear procedure for the variational problem solution via Bernstein polynomials technique. The variational problems are solved by means of the direct method using the Bernstein polynomials and reduced to the solution of algebraic equations. The property of Bernstein polynomials is fully applied to shorten the calculation process in the task. Three illustrative examples are included.

Keywords: Bernstein polynomials, variational problem, direct method

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I. Introduction

Bernstein polynomials have been applied recently to solve some linear and nonlinear differential equations by Bhatta and Bhatti (Bhatta & Bhatti (2006)), Bhatti and Bracken (Bhatti & Bracken (2007)). The useful mathematical tool Bernstein polynomials are extensively used on signal processing in communications (Caglar & Akansu (1993)) and physics research. To the authors' knowledge, however, this powerful tool has not been used for solving variational problems.

In literature, the well-known direct method of Ritz and Galerkin has been applied to the variational problem by Schecher (Schechter (1967)), and the gradient method has been taken successfully to complete some similar research (Miele, Tietze, & Levy (1972), Miele (1975)). Many authors (Chang & Wang (1983), Hwang & Shih (1983), Horng & Chou (1985), Razzaghi & Marzban (2000), Razzaghi & Yousefi (2000), Razzaghi & Ordokhani (2001), Babolian, Mokhtari, & Salmani (2007), Ordokhani (2011), Razzaghi, Ordokhani, & Haddadi (2012), Jarczewska, Glabisz, & Zielichowski-Haber (2015), Rahman Jaber (2015), Zarebnia & Barandak Imcheh (2016), Hassan Ouda (2018)) have tried various transform methods to overcome these difficulties in the problem of extremization of a functional systems. The fundamental idea of a direct method for solving variational problems is to convert the problem of extremization of a functional into one which involves a finite number of variables.

This paper focuses on the solution of variational problems via Bernstein transform by taking advantage of the nice properties of Bernstein polynomials. First, it introduces Bernstein polynomials, which is tutorial in nature; then presents a direct method for solving variational problems via Bernstein polynomials. The procedure involves (i) assuming the admissible functions by Bernstein polynomials with coefficients to be determined; (ii) establishing an operational matrix for performing integration; (iii) finding the necessary condition for extremization; and (iv) solving for the algebraic equation obtained from the previous steps to evaluate Bernstein coefficients. Because of the local property of the powerful Bernstein polynomials, the new direct method is simpler in reasoning as well as in calculation.

As for Laguerre polynomials (Hwang & Shih (1983)), Legendre polynomials (Chang & Wang (1983)), and Chebyshev polynomials (Horng & Chou (1985)), the calculation procedures are usually too tedious, and some recursive formulae are still waiting for developments. These polynomials are unable to compare with Bernstein expansion with respect to computation time and data storage requirements. However, Bernstein's method does achieve higher accuracy than these polynomials in the most of variational problems.

II. Some properties of Bernstein polynomials

A. Polynomial basis

The B-polynomial of n-th degree are defined on the interval $[0,1]$ as (Bhatti et al. (2007)).

$$B_{i,n}(t)=\binom{n}{i}t^i\frac{(R-t)^{n-i}}{R^n},0\le i\le n, \tag{1}$$

for $i=0,1,\dots,n$, where the binomial coefficients are the combinations given by

$$\binom{n}{i}=\frac{n!}{i!\,(n-i)!}, \tag{2}$$

and R is the maximum such that the polynomials are defined to form a complete basis over the interval $[0,R]$. These polynomials are written down handily. As i increases by 1, the exponent on the t-term increases by 1 while the exponent on the $(R-t)$-term decreases by 1. The recursive definition of the B-polynomials over this interval is generated below:

$$B_{i,n}(t)=\frac{(R-t)}{R}B_{i,n-1}(t)+\frac{t}{R}B_{i-1,n-1}(t). \tag{3}$$

B. Function approximation

Any function $y(t)$ which is square integrable in the interval $[0,1]$ can be expanded in Bernstein polynomials with an infinite number of terms

$$y(t)=\sum_{i=0}^{\infty}c_i\ b_i(t),\ t\in[0,1]. \tag{4}$$

Usually, the polynomial expansion (4) contains an infinite number of terms for a smooth $y(t)$. If $y(t)$ is a continuous function, then the sum in (4) may be terminated after m terms, that is

$$y(t) \approx \sum_{i=0}^{m-1} c_i\, b_i(t) = \boldsymbol{c}_{(m)}^T \boldsymbol{b}_{(m)}(t) \triangleq y^*(t),\ t \in [0,1], \tag{5}$$

$$\boldsymbol{c}_{(m)} \triangleq [\,c_0\ \ c_1\ \ \cdots\ \ c_{m-1}]^T, \tag{6}$$

$$\boldsymbol{b}_{(m)}(t) \triangleq [\,b_0(t)\ \ b_1(t)\ \ \cdots\ \ b_{m-1}(t)]^T, \tag{7}$$

where "T" indicates transposition, the subscript m in the parentheses denotes their dimensions, $y^*(t)$ denotes the truncated sum. Let us define the m-square Bernstein matrix as

$$B_{(m\times m)} \triangleq [\,\boldsymbol{b}_{(m)}(\frac{1}{2m})\ \ \boldsymbol{b}_{(m)}(\frac{3}{2m})\ \ \cdots\ \ \boldsymbol{b}_{(m)}(\frac{2m-1}{2m})]. \tag{8}$$

Substituting $t = \frac{1}{2m}, \frac{3}{2m}, \cdots, \frac{2m-1}{2m}$ into (6) yields

$$y_{(m)}^* \triangleq [\,y^*(\frac{1}{2m})\ \ y^*(\frac{3}{2m})\ \ \cdots\ \ y^*(\frac{2m-1}{2m})] = \boldsymbol{c}_{(m)}^T B_{(m\times m)}. \tag{9}$$

It is obvious that

$$\boldsymbol{c}_{(m)}^T = y_{(m)}^* B_{(m\times m)}^{-1}. \tag{10}$$

Equation (10) is called the forward transform, which transforms the time function $y_{(m)}^*$ into the coefficient vector $\boldsymbol{c}_{(m)}^T$, and (9) is called the *inverse transform*, which recovers $y_{(m)}^*$ from $\boldsymbol{c}_{(m)}^T$.

C. Integration of Bernstein polynomials

In Bernstein polynomials analysis for a dynamic system, all functions need to be transformed into Bernstein polynomials. The integration of Bernstein polynomials can be expanded into Bernstein polynomials with Bernstein coefficient matrix P.

$$\int_0^t \boldsymbol{b}_{(m)}(\tau)d\tau \approx P_{(m\times m)}\boldsymbol{b}_{(m)}(t),\ t \in [0,1], \tag{11}$$

where the m-square matrix P defined below is called the operational matrix of the Bernstein integral.

D. Multiplication of Bernstein polynomials

In the study of time-varying system via Bernstein polynomials, it is usually necessary to evaluate $\boldsymbol{b}_{(m)}(t)\boldsymbol{b}_{(m)}^T(t)$. Let $\boldsymbol{b}_{(m)}(t)\boldsymbol{b}_{(m)}^T(t) \overset{\Delta}{=} M_{(m\times m)}(t)$ which is called the product matrix of Bernstein polynomials. That is,

$$M_{(m\times m)}(t) \overset{\Delta}{=} \begin{bmatrix} b_0b_0 & b_0b_1 & b_0b_2 & b_0b_3 & \cdots & b_0b_{m-1} \\ b_1b_0 & b_1b_1 & b_1b_2 & b_1b_3 & \cdots & b_1b_{m-1} \\ b_2b_0 & b_2b_1 & b_2b_2 & b_2b_3 & \cdots & b_2b_{m-1} \\ b_3b_0 & b_3b_1 & b_3b_2 & b_3b_3 & \cdots & b_3b_{m-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ b_{m-1}b_0 & b_{m-1}b_1 & b_{m-1}b_2 & b_{m-1}b_3 & \cdots & b_{m-1}b_{m-1} \end{bmatrix}. \tag{12}$$

The matrix $M_{(m\times m)}(t)$ satisfies

$$M_{(m\times m)}(t)\boldsymbol{c}_{(m)} = C_{(m\times m)}\boldsymbol{b}_{(m)}(t), \tag{13}$$

where $\boldsymbol{c}_{(m)}$ is defined as (6).

$$\int_0^1 \boldsymbol{b}_{(m)}(\tau)\boldsymbol{b}_{(m)}^T(\tau)\, d\tau = \int_0^1 M_{(m\times m)}(\tau)\, d\tau \overset{\Delta}{=} K_{(m\times m)}. \tag{14}$$

Eq. (14) is very important for solving variational problems.

III. Direct method for simple variational problem

The regular method for solving the extremization problem of a functional:

$$J = \int_0^1 F[t, \boldsymbol{x}(t), \dot{\boldsymbol{x}}(t)]\, dt \tag{15}$$

is through the Euler equation

$$F_x - (d/dt)F_{\dot{x}} = 0. \tag{16}$$

However, the differential equation so obtained can be integrated easily only in exceptional cases. Therefore, many direct methods have been developed. Ritz's and Galerkin's methods are well known (Gelfand & Fomin (1963), Brewster (1958)). This paper mainly uses Bernstein polynomials to establish the direct method for variational problems.

Unlike other direct methods, beginning with the assumption of the variable itself, the method we have developed starts with the rate variable. In other words, we assume the rate variable $\dot{\boldsymbol{x}}(t)$ as Bernstein polynomials whose coefficients are to be determined,

$$\dot{\boldsymbol{x}}(t)=\sum_{i=0}^{\infty} c_i\ b_i(t). \tag{17}$$

Taking finite terms as an approximation, we have

$$\dot{\boldsymbol{x}}(t)\approx \boldsymbol{c}_{(m)}^T \boldsymbol{b}_{(m)}(t). \tag{18}$$

Integrating Eq. (18) from 0 to t and using Eq. (11), the variable $\boldsymbol{x}(t)$ can be expressed as

$$\begin{aligned}\boldsymbol{x}(t)=\int_0^t \dot{\boldsymbol{x}}(\tau)\ d\tau+\boldsymbol{x}(0)\approx \boldsymbol{c}_{(m)}^T\int_0^t \boldsymbol{h}_{(m)}(\tau)\ d\tau+\boldsymbol{x}(0)\\ \approx \boldsymbol{c}_{(m)}^T P_{(m\times m)}\boldsymbol{b}_{(m)}(t)+\boldsymbol{x}(0).\end{aligned} \tag{19}$$

The other terms in the functional of Eq. (15) are known functions of the independent variable t and can be expanded into Bernstein polynomials through substitution, we finally have

$$J=J(c_0,c_1,\dots,c_{m-1}). \tag{20}$$

The original extremization of a functional problem shown in Eq. (15) becomes the extremization of a function of a finite set of variables in Eq. (20).

Taking partial derivatives of J with respect to c_i, and setting them equal to zero, we obtain

$$\partial J/\partial c_i=0\ \ (i=0,1,\dots,m-1). \tag{21}$$

Solving for c_i, and substituting into Eq. (19), we have the result.

We note that the above proposed method implies Euler's direct method of finite difference and is similar to Ritz's method using power series and Fourier series; but considering (i) the property of Bernstein polynomials and (ii) the product property of P shown in Eq. (11) and the operational property of P itself, we can claim that the new direct method via Bernstein polynomials is much simpler and more powerful than any previous direct methods.

Let us establish the detailed procedure via several classical problems.

IV. Illustrative examples

A. First-order functional extremal with two fixed boundary conditions

Find the extremal of the following functional:

$$J=\int_{0}^{1} \dot{\boldsymbol{x}}^{2}(t)+t\dot{\boldsymbol{x}}(t)\ dt \tag{22}$$

The boundary conditions are the initial condition and the final condition,

$$\boldsymbol{x}(0)=0, \tag{23}$$

$$\boldsymbol{x}(1)=1/4. \tag{24}$$

For solving this problem by the Bernstein direct method, we assume that $\dot{\boldsymbol{x}}(t)$ can be expanded in terms of Bernstein polynomials as Eq. (18). Here we let $m=8$ for clarity in presentation; more accurate results can be obtained by using a larger m. There is a variable t involved in Eq. (22) explicitly; it can be expanded into Bernstein polynomials over the time interval $[0,1]$,

$$t\approx \boldsymbol{d}_{(m)}^{T}\boldsymbol{b}_{(m)}(t). \tag{25}$$

Substituting Eqs. (18) and (35) into Eq. (22), we have

$$\begin{aligned} J\approx\int_{0}^{1} & [\boldsymbol{c}_{(m)}^{T}\boldsymbol{b}_{(m)}(t)\boldsymbol{b}_{(m)}^{T}(t)\boldsymbol{c}_{(m)} \\ & +\boldsymbol{c}_{(m)}^{T}\boldsymbol{b}_{(m)}(t)\boldsymbol{b}_{(m)}^{T}(t)\boldsymbol{d}_{(m)}]\ dt. \end{aligned} \tag{26}$$

However, the vector function $\boldsymbol{b}_{(m)}(t)$ has a particular property as Eq. (14) due to the orthogonality of Bernstein polynomials. Using Eq. (14), Eq. (26) simply becomes

$$J\approx \boldsymbol{c}_{(m)}^{T}K_{(m\times m)}\boldsymbol{c}_{(m)}+\boldsymbol{c}_{(m)}^{T}K_{(m\times m)}\boldsymbol{d}_{(m)}. \tag{27}$$

For the initial boundary condition, substituting Eq. (23) into Eq. (19), yields

$$\boldsymbol{x}(t)\approx \boldsymbol{c}_{(m)}^{T}\int_{0}^{t} \boldsymbol{b}_{(m)}(\tau)\ d\tau+0\approx \boldsymbol{c}_{(m)}^{T}P_{(m\times m)}\boldsymbol{b}_{(m)}(t) \tag{28}$$

For the final boundary condition, substituting Eq. (24) into Eq. (28), yields

$$\boldsymbol{x}(1) = \boldsymbol{c}_{(m)}^T \int_0^1 \boldsymbol{b}_{(m)}(\tau)\, d\tau = 1/4. \tag{29}$$

It is interesting to note that the definite integral of b_i from 0 to 1 is equal to 1/8 for $m = 8$; or

$$\int_0^1 b_i(\tau)\, d\tau = 1/8,\ i = 0,2,\dots,7. \tag{30}$$

Substituting Eq. (30) into Eq. (29) simply gives

$$\begin{aligned}\boldsymbol{c}_{(m)}^T[1,1,\ \cdots\ ,1]^T &= c_0 + c_1 + c_2 + c_3 + c_4 + c_5 + c_6 + c_7 \\ &= 2,\end{aligned} \tag{31}$$

$$c_0 = 2 - c_1 - c_2 - c_3 - c_4 - c_5 - c_6 - c_7. \tag{32}$$

c_0 is found with little effort. This information should be substituted into Eq. (27) also; we then have

$$\begin{aligned} J \approx\ & (7c_1^2)/195 + (28c_2^2)/429 + (1043c_3^2)/12870 + (574c_4^2)/6435 + (203c_5^2)/2145 \\ & +(658c_6^2)/6435 + (7c_1c_2)/78 + (371c_1c_3)/4290 + (56c_1c_4)/715 \\ & +(14c_1c_5)/195 + (1757c_1c_6)/25740 \\ & + (1723c_1c_7)/25740 + \cdots + 53/180. \end{aligned} \tag{33}$$

For extremization, we take the partical derivatives of J with respect to c_i, $i = 1,2,\dots,7$, for $m = 8$ and set it equal to zero

$$\partial J/\partial c_1 = 0,\ \partial J/\partial c_2 = 0,\ \dots\ ,\partial J/\partial c_7 = 0. \tag{34}$$

Therefore,

$$\dot{\boldsymbol{x}}(t) \approx [0.5000, 0.4286, 0.3571, \cdots, -0.0000]\boldsymbol{b}_{(m)}(t). \tag{35}$$

And $\boldsymbol{x}(t)$ is obtained from Eq. (28),

$$\boldsymbol{x}(t) \approx [0.0000, 0.0714, 0.1310, \cdots, 0.2500]\boldsymbol{b}_{(m)}(t). \tag{36}$$

If the Euler equation is used for the analytic solution, the answer should be

$$\dot{\boldsymbol{x}}(t) = (1/2)(1 - t), \tag{37}$$

$$\boldsymbol{x}(t) = (t/2)(1 - t/2), \tag{38}$$

respectively. The comparison of the solutions via Euler's analytic method and via Bernstein's direct method is shown in Figure 1 and Table 2. It is seen that even when $m = 8$, the Bernstein direct method is quite satisfactory.

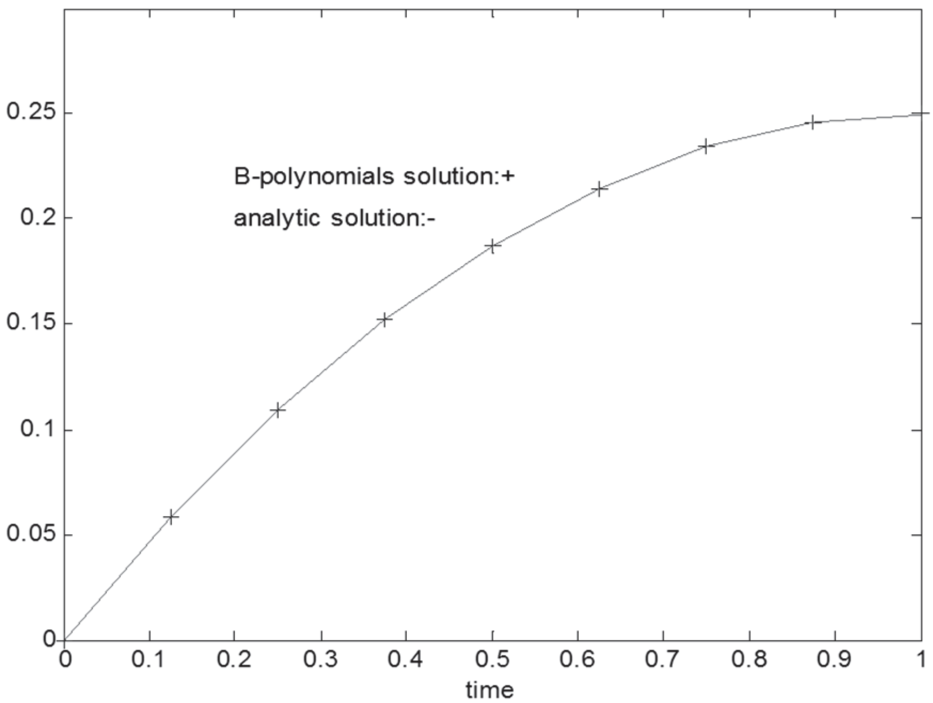

Figure 1. Functional extremal with two fixed boundary conditions.

Table 2. Bernstein and analytic solutions of the state variable $x(t)$

t	Bernstein solution	analytic solution
0	0	0
0.125	0.0586	0.0586
0.250	0.1094	0.1094
0.375	0.1523	0.1523
0.500	0.1875	0.1875
0.625	0.2148	0.2148
0.750	0.2344	0.2344
0.875	0.2461	0.2461
1	0.2500	0.2500

B. First-order functional extremal with a fixed boundary condition and a moving boundary condition

Let us consider the same functional extremal of Eq. (22) but with unspecified $x(1)$, namely,

$$\boldsymbol{x}(0) = 0, \tag{39}$$

$$\boldsymbol{x}(1) \text{ unspecified.} \tag{40}$$

The another condition in addition to Eq. (40) may be found from $F[t, \boldsymbol{x}(t), \dot{\boldsymbol{x}}(t)]$,

$$F_{\dot{\boldsymbol{x}}}|_{t=1} = 0, \ \ \dot{\boldsymbol{x}}(1) = -1/2. \tag{41}$$

Substituting $t = 1$ into Eq. (18) for $m = 8$, we have

$$\dot{\boldsymbol{x}}(1) = \boldsymbol{c}_{(m)}^T \boldsymbol{b}_{(m)}(1) = c_7 = -1/2. \tag{42}$$

Of course, the ramp function t in the functional F can still be expressed as Eq. (25). Substituting Eqs. (18), (25), and (42) into Eq. (22), we have

$$J \approx \boldsymbol{c}_{(m)}^T K_{(m\times m)} \boldsymbol{c}_{(m)} + \boldsymbol{c}_{(m)}^T K_{(m\times m)} \boldsymbol{d}_{(m)}$$

$$= c_0^2/15 + (7c_1^2)/195 + (21c_2^2)/715 + (35c_3^2)/1287 + (35c_4^2)/1287 + (21c_5^2)/715$$

$$+(7c_6^2)/195 + \cdots - 7/180. \tag{43}$$

J is extremized by setting its partial derivatives as Eq. (34). Namely,

$$\boldsymbol{c}_{(m)}^T = [-0.0000, -0.0714, -0.1429, \cdots, -0.5000]. \tag{44}$$

Therefore,

$$\dot{\boldsymbol{x}}(t) \approx [-0.0000, -0.0714, -0.1429, \cdots, -0.5000] \boldsymbol{b}_{(m)}(t). \tag{45}$$

$$\boldsymbol{x}(t) \approx [-0.0000, 0.0000, -0.0119, \cdots, -0.2500] \boldsymbol{b}_{(m)}(t). \tag{46}$$

Analytic solution via Euler's equation is

$$\dot{\boldsymbol{x}}(t) = -t/2, \tag{47}$$

$$\boldsymbol{x}(t) = -t^2/4. \tag{48}$$

The following tabulated values in Table 2 and Figure 2 are set up for comparison.

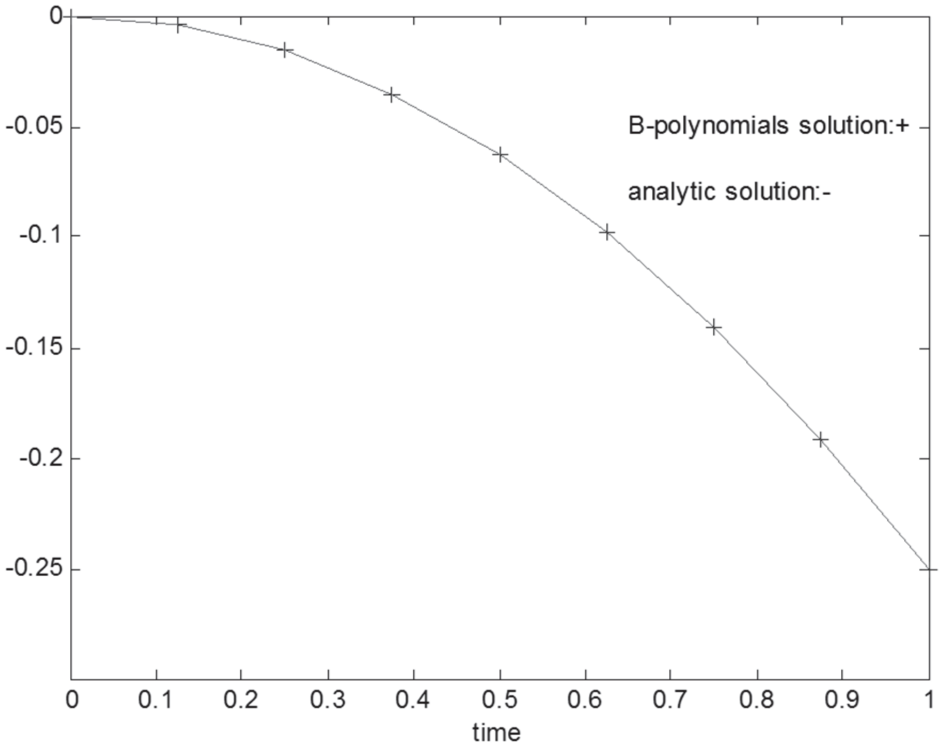

Figure 2. Functional extremal with a fixed boundary condition and a moving boundary condition.

Table 2. Bernstein and analytic solutions of the state variable $x(t)$

t	Bernstein solution	analytic solution
0	0	0
0.125	-0.0039	-0.0039
0.250	-0.0156	-0.0156
0.375	-0.0352	-0.0352
0.500	-0.0625	-0.0625
0.625	-0.0977	-0.0977
0.750	-0.1406	-0.1406
0.875	-0.1914	-0.1914
1	-0.2500	-0.2500

C. Second-order functional extremal with two fixed boundary conditions and two moving boundary conditions

Suppose we want to find the following functional extremal

$$J = \int_0^1 [\ddot{\boldsymbol{x}}^2(t)/2 + (4-4t)\dot{\boldsymbol{x}}(t)]\ dt, \tag{49}$$

$$\boldsymbol{x}(0) = 0, \tag{50}$$

$$\dot{\boldsymbol{x}}(0) = 0, \tag{51}$$

$$\boldsymbol{x}(1),\ \dot{\boldsymbol{x}}(1)\ \text{ unspecified.} \tag{52}$$

The natural boundary conditions are found from follows.

$$F_{\dot{x}} - d/dt(F_{\ddot{x}})|_{t=1} = 0,\ (4-4t) - \dddot{x}\,|_{t=1} = 0,\ \dddot{x}(1) = 0, \tag{53}$$

$$F_{\ddot{x}}|_{t=1} = 0,\ \ddot{\boldsymbol{x}}(1) = 0. \tag{54}$$

Let us expand $\dddot{x}(t)$ into Bernstein polynomials with m terms,

$$\dddot{x}(t) \approx \sum_{i=0}^{m} c_i\, b_i(t) = \boldsymbol{c}_{(m)}^T \boldsymbol{b}_{(m)}(t). \tag{55}$$

Integrating $\dddot{x}(t)$ and applying Eq. (11) for $m = 8,\ t \in [0,1]$ yield $\ddot{\boldsymbol{x}}(t)$ and $\dot{\boldsymbol{x}}(t)$.

$$\begin{aligned}\ddot{\boldsymbol{x}}(t) \approx \boldsymbol{c}^T \int_0^t \boldsymbol{b}(\tau)\ d\tau + \ddot{\boldsymbol{x}}(0) &\approx \boldsymbol{c}^T P \boldsymbol{b}(t) + \ddot{\boldsymbol{x}}(0) \\ &= \{\boldsymbol{c}^T P + \ddot{\boldsymbol{x}}(0)[1,1,\dots,1]B^{-1}\}\boldsymbol{b}(t),\end{aligned} \tag{56}$$

$$\dot{\boldsymbol{x}}(t) \approx \{\boldsymbol{c}^T P + \ddot{\boldsymbol{x}}(0)[1,1,\dots,1]B^{-1}\}\int_0^t \boldsymbol{b}(\tau)\ d\tau + \dot{\boldsymbol{x}}(0),\ (\dot{\boldsymbol{x}}(0) = 0)$$

$$\approx [\boldsymbol{c}^T P + \boldsymbol{f}^T B^{-1}\ddot{\boldsymbol{x}}(0)]P\boldsymbol{b}(t), \tag{57}$$

where vector $\boldsymbol{f}$ is defined as

$$\boldsymbol{f} = [1,1,1,1,1,1,1,1,1]^T. \tag{58}$$

The natural conditions of Eq. (52), Eq. (54) require

$$\dddot{x}(1) \approx \boldsymbol{c}^T \boldsymbol{b}(1) \approx 0,\ c_7 = 0, \tag{59}$$

$$\ddot{\boldsymbol{x}}(1) \approx \boldsymbol{c}^T \int_0^1 \boldsymbol{b}(\tau)\, d\tau + \ddot{\boldsymbol{x}}(0) = 1/8\boldsymbol{c}^T \boldsymbol{f} + \ddot{\boldsymbol{x}}(0) \approx 0, \tag{60}$$

$$\ddot{\boldsymbol{x}}(0) \approx -1/8\boldsymbol{c}^T \boldsymbol{f} = -1/8\boldsymbol{c}^T \boldsymbol{f} B^{-1}\boldsymbol{b}(t) = \boldsymbol{c}^T Q \boldsymbol{b}(t), \tag{61}$$

where matrix Q is defined as

$$Q \triangleq -1/8\boldsymbol{f}B^{-1}. \tag{62}$$

For convenience of operation in vectors, Eqs. (56) and (57) may be rewritten as

$$\ddot{\boldsymbol{x}}(t) = \boldsymbol{c}^T(P+Q)\boldsymbol{b}(t), \tag{63}$$

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{c}^T(P+Q)P\boldsymbol{b}(t). \tag{64}$$

The explicit time function in the functional may be expanded into Bernstein polynomials

$$4-4t = \boldsymbol{g}^T\boldsymbol{b}(t), \tag{65}$$

where the vector $\boldsymbol{g}$ is defined as

$$\boldsymbol{g} = [4.0000, 3.4286, 2.8571, 2.2857, 1.7143, 1.1429, 0.5714, 0]^T. \tag{66}$$

The integrand of J may be put into quadratic form

$$F(t,\boldsymbol{x},\dot{\boldsymbol{x}},\ddot{\boldsymbol{x}}) = \ddot{\boldsymbol{x}}^2(t)/2 + (4-4t)\dot{\boldsymbol{x}}(t)$$

$$\approx (1/2)\boldsymbol{c}^T(P+Q)\boldsymbol{b}(t)\boldsymbol{b}^T(t)(P^T+Q)\boldsymbol{c} + \boldsymbol{c}^T(P+Q)P\boldsymbol{b}(t)\boldsymbol{b}^T(t)\boldsymbol{g}. \tag{67}$$

After integration, J becomes a function of $\boldsymbol{c}$

$$J \approx (1/2)\boldsymbol{c}^T(P+Q)K(P^T+Q)\boldsymbol{c} + [\boldsymbol{c}^T(P+Q)P]K\boldsymbol{g}. \tag{68}$$

We wish to minimize J with respect to $\boldsymbol{c}$ subject to the constraint Eq. (59). Langrange multiplier λ may be applied here to take care of the constraint. Let

$$J^* = J + \lambda\boldsymbol{c}^T\boldsymbol{b}(1)$$

$$\approx (1/2)\boldsymbol{c}^T(P+Q)K(P^T+Q)\boldsymbol{c} + \boldsymbol{c}^T(P+Q)PK\boldsymbol{g} + \lambda\boldsymbol{c}^T\boldsymbol{b}(1). \tag{69}$$

Taking partial derivatives of J^* with respect to $\boldsymbol{c}$, we have

$$\partial J^*/\partial \boldsymbol{c}^T = \boldsymbol{0} \approx (P+Q)K(P^T+Q)\boldsymbol{c} + (P+Q)PK\boldsymbol{g} + \lambda \boldsymbol{b}(1). \tag{70}$$

Eqs. (59) and (70) may be combined

$$\begin{bmatrix} (P_{(m\times m)}+Q_{(m\times m)})K_{(m\times m)}(P^T_{(m\times m)}+Q_{(m\times m)}) & \boldsymbol{b}_{(m)}(1) \\ \boldsymbol{b}^T_{(m)}(1) & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{c}_{(m)} \\ \lambda \end{bmatrix} = \begin{bmatrix} -(P_{(m\times m)}+Q_{(m\times m)})P_{(m\times m)}K_{(m\times m)}\boldsymbol{g}_{(m)} \\ 0 \end{bmatrix}. \tag{71}$$

After $\boldsymbol{c}$ is solved from the linear algebraic equation of Eq. (71), $\ddot{\boldsymbol{x}}(t)$, and $\dot{\boldsymbol{x}}(t)$ may be evaluated with Eqs. (63) and (64). Then,

$$\boldsymbol{x}(t) \approx [-0.0000, 0.0005, -0.0533, \cdots, -0.5007]\boldsymbol{b}(t) \tag{72}$$

The analytic solution via Euler equation is

$$\ddot{\boldsymbol{x}}(t) = -2t^2 + 4t - 2. \tag{73}$$

$$\dot{\boldsymbol{x}}(t) = -(2/3)t^3 + 2t^2 - 2t, \tag{74}$$

$$\boldsymbol{x}(t) = -t^4/6 + (2/3)t^3 - t^2. \tag{75}$$

The comparison between the Bernstein solution and the analytic solution is shown in Figure 3 and Table 3 for $m = 8$, which confirms that with respect to numerical solutions, the Bernstein approach gives almost the same as the analytic method. Better approximation is expected by choosing a larger value of m.

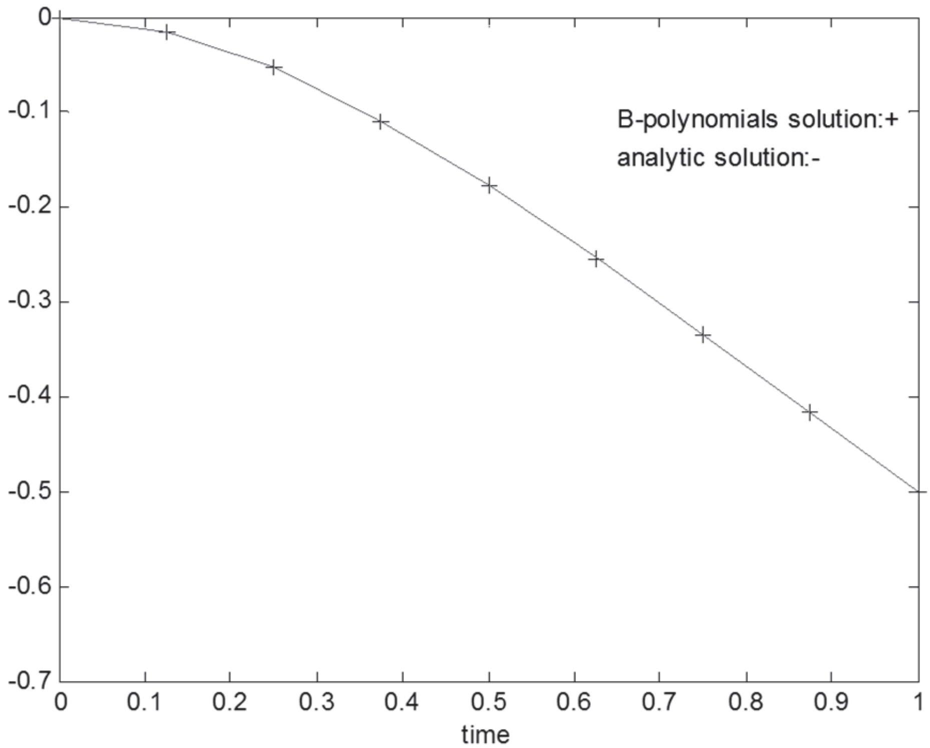

Figure 3. Functional extremal with two fixed boundary conditions and two moving boundary conditions.

Table 3. Bernstein and analytic solutions of the state variable $x(t)$

t	Bernstein solution	analytic solution
0	0	0
0.125	-0.0138	-0.0144
0.250	-0.0518	-0.0527
0.375	-0.1075	-0.1088
0.500	-0.1758	-0.1771
0.625	-0.2521	-0.2533
0.750	-0.3330	-0.3340
0.875	-0.4161	-0.4167
1	-0.4999	-0.5000

V. Conclusions

After briefly reviewing the Bernstein polynomials techniques, we form an operational matrix for performing integrations in Bernstein polynomials analysis. A direct method of variation is established by using Bernstein polynomials. A simple extremization problem and a variational problem are completely solved step by step by the new proposed procedure. It is believed that the approach is more powerful either than Ritz's or Euler's direct methods for solving variational problems.

The method of using the Bernstein polynomials to solve variational problems reduces variational problems to the solution of algebraic equations, and so the calculation is straightforward and digital computer oriented.

Some fundamental properties on Bernstein polynomials such as Eqs. (11) and (14) have been derived, and some effective algorithms have been applied to solve the rather difficult variational problems successfully. The main contributions should be ascribed to the Bernstein polynomials. We are fully confident of the future development for the Bernstein transform method, since the sound base has been established.

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